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Crystal Analysis of type C Stanley Symmetric Functions

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Abstract. Combining results of T. K. Lam and J. Stembridge, the type C Stanley symmetric function $F_w^C(\mathbf{x})$, indexed by an element w in the type C Coxeter group, has a nonnegative integer expansion in terms of Schur functions. We provide a crystal theoretic explanation of this fact and give an explicit combinatorial description of the coefficients in the Schur expansion in terms of highest weight crystal elements.

Keywords: Stanley symmetric functions, crystal bases, Kraśkiewicz insertion, mixed Haiman insertion, unimodal tableaux, primed tableaux

1 Introduction

Schubert polynomials of types B and C were independently introduced by Billey and Haiman [1] and Fomin and Kirillov [6]. Stanley symmetric functions [15] are stable limits of Schubert polynomials, designed to study properties of reduced words of Coxeter group elements. In his Ph.D. thesis, T. K. Lam [12] studied properties of Stanley symmetric functions of types B (and similarly C) and D. In particular he showed, using Kraśkiewicz insertion [10, 11], that the type B Stanley symmetric functions have a positive integer expansion in terms of P-Schur functions. On the other hand, Stembridge [16] proved that the P-Schur functions expand positively in terms of Schur functions. Combining these two results, it follows that Stanley symmetric functions of type B (and similarly type C) have a positive integer expansion in terms of Schur functions.

Schur functions $s_\lambda(\mathbf{x})$, indexed by partitions λ, are ubiquitous in combinatorics and representation theory. They are the characters of the symmetric group and can also be interpreted as characters of type A crystals. In [13], this was exploited to provide a combinatorial interpretation in terms of highest weight crystal elements of the coefficients in the Schur expansion of Stanley symmetric functions in type A. In this paper, we carry out a crystal analysis of the Stanley symmetric functions $F_w^C(\mathbf{x})$ of type C, indexed by a Coxeter group element w. In particular, we use Kraśkiewicz insertion [10, 11] and Haiman's

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mixed insertion [7] to find a crystal structure on primed tableaux, which in turn implies a crystal structure $\mathcal{B}_w$ on signed unimodal factorizations of w for which $F^C_w(\mathbf{x})$ is a character. Moreover, we present a type A crystal isomorphism $\Phi\colon \mathcal{B}_w \to \bigoplus_\lambda \mathcal{B}_\lambda^{\oplus g_{w\lambda}}$ for some combinatorially defined nonnegative integer coefficients $g_{w\lambda}$; here $\mathcal{B}_\lambda$ is the type A highest weight crystal of highest weight λ . This implies the desired decomposition $F^C_w(\mathbf{x}) = \sum_\lambda g_{w\lambda} s_\lambda(\mathbf{x})$ (see Corollary 23) and similarly for type B.

In Section 2, we review type C Stanley symmetric functions and type A crystals. In Section 3 we describe our crystal isomorphism by combining a slight generalization of the Kraśkiewicz insertion [10, 11] and Haiman's mixed insertion [7]. The main result regarding the crystal structure under Haiman's mixed insertion is stated in Theorem 18. The combinatorial interpretation of the coefficients $g_{w\lambda}$ is given in Corollary 23.

2 Background

2.1 Type C Stanley symmetric functions

The **Coxeter group** W_C of type C_n (or B_n), also known as the hyperoctahedral group or the group of signed permutations, is a finite group generated by $\{s_0, s_1, \ldots, s_{n-1}\}$ subject to the quadratic relations $s_i^2 = 1$ for all $i \in I = \{0, 1, \ldots, n-1\}$, the commutation relations $s_i s_j = s_j s_i$ provided $|i - j| > 1$, and the braid relations $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ for all $i > 0$ and $s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0$.

It is often convenient to write down an element of a Coxeter group as a sequence of indices of s_i in the product representation of the element. For example, the element $w = s_2 s_1 s_2 s_1 s_0 s_1 s_0 s_1$ is represented by the word $\mathbf{w} = 2120101$. A word of shortest length ℓ is referred to as a **reduced word** and $\ell(w) := \ell$ is referred as the length of w. The set of all reduced words of the element w is denoted by $R(w)$.

We say that a reduced word $a_1 a_2 \ldots a_\ell$ is **unimodal** if there exists an index v, such that $a_1 > a_2 > \cdots > a_v < a_{v+1} < \cdots < a_\ell$. Consider a reduced word $\mathbf{a} = a_1 a_2 \ldots a_{\ell(w)}$ of a Coxeter group element w. A **unimodal factorization** of $\mathbf{a}$ is a factorization $\mathbf{A} = (a_1 \ldots a_{\ell_1})(a_{\ell_1+1} \ldots a_{\ell_2}) \cdots (a_{\ell_r+1} \ldots a_L)$ such that each factor $(a_{\ell_i+1} \ldots a_{\ell_{i+1}})$ is unimodal. Factors can be empty.

For a fixed Coxeter group element w, consider all reduced words $R(w)$, and denote the set of all unimodal factorizations for reduced words in $R(w)$ as $U(w)$. Given a factorization $\mathbf{A} \in U(w)$, define the **weight** of a factorization $\mathrm{wt}(\mathbf{A})$ to be the vector consisting of the number of elements in each factor. Denote by $\mathrm{nz}(\mathbf{A})$ the number of non-empty factors of $\mathbf{A}$.

Example 1. For the factorization $\mathbf{A} = (2102)()(10) \in U(s_2 s_1 s_2 s_0 s_1 s_0)$, we have $\mathrm{wt}(\mathbf{A}) = (4, 0, 2)$ and $\mathrm{nz}(\mathbf{A}) = 2$.

Following [1, 6, 12], the **type C Stanley symmetric function** associated to $w \in W_C$ is defined as

$$F_w^C(\mathbf{x}) = \sum_{\mathbf{A} \in U(w)} 2^{\mathrm{nz}(\mathbf{A})} \mathbf{x}^{\mathrm{wt}(\mathbf{A})}. \tag{2.1}$$

Here $\mathbf{x} = (x_1, x_2, x_3, \ldots)$ and $\mathbf{x}^{\mathbf{v}} = x_1^{v_1} x_2^{v_2} x_3^{v_3} \cdots$. It is not obvious from the definition why the above functions are symmetric. We refer reader to [2], where this fact follows easily from an alternative definition. **Type B Stanley symmetric functions** are also labeled by $w \in W_C$ given by $F_w^B(\mathbf{x}) = 2^{-o(w)} F_w^C(\mathbf{x})$, where $o(w)$ is the number of zeroes in a reduced word for w.

2.2 Type A crystal of words

Crystal bases [8] play an important role in many areas of mathematics. For example, they make it possible to analyze representation theoretic questions using combinatorial tools. Here we only review the crystal of words in type A_n and refer the reader for more background on crystals to [3].

Consider the set of words $\mathcal{B}_n^h$ of length h in the alphabet $\{1, 2, \ldots, n+1\}$. We impose a crystal structure on $\mathcal{B}_n^h$ by defining lowering operators f_i and raising operators e_i for $1 \leqslant i \leqslant n$ and a weight function. The weight of $\mathbf{b} \in \mathcal{B}_n^h$ is the tuple $\mathrm{wt}(\mathbf{b}) = (a_1, \ldots, a_{n+1})$, where a_i is the number of letters i in $\mathbf{b}$. The crystal operators f_i and e_i only depend on the letters i and $i+1$ in $\mathbf{b}$. Consider the subword $\mathbf{b}^{\{i,i+1\}}$ of $\mathbf{b}$ consisting only of the letters i and $i+1$. Successively bracket any adjacent pairs $(i+1)i$ and remove these pairs from the word. The resulting word is of the form $i^a(i+1)^b$ with $a, b \geqslant 0$. Then f_i changes this subword within $\mathbf{b}$ to $i^{a-1}(i+1)^{b+1}$ if $a > 0$ leaving all other letters unchanged and otherwise annihilates $\mathbf{b}$. The operator e_i changes this subword within $\mathbf{b}$ to $i^{a+1}(i+1)^{b-1}$ if $b > 0$ leaving all other letters unchanged and otherwise annihilates $\mathbf{b}$. We call an element $\mathbf{b} \in \mathcal{B}_n^h$ **highest weight** if $e_i(\mathbf{b}) = \mathbf{0}$ for all $1 \leqslant i \leqslant n$.

Theorem 2. *[9] A word $\mathbf{b} = b_1 \ldots b_h \in \mathcal{B}_n^h$ is highest weight if and only if it is a Yamanouchi word. That is, for any index k with $1 \leqslant k \leqslant h$ the weight of a subword $b_k b_{k+1} \ldots b_h$ is a partition.*

Two crystals $\mathcal{B}$ and $\mathcal{C}$ are said to be **isomorphic** if there exists a bijective map $\Phi \colon \mathcal{B} \to \mathcal{C}$ that preserves the weight function and commutes with the crystal operators e_i and f_i. A **connected component** X of a crystal is a set of elements where for any two $\mathbf{b}, \mathbf{c} \in X$ one can reach $\mathbf{c}$ from $\mathbf{b}$ by applying a sequence of f_i and e_i.

Theorem 3. *[9] Each connected component of $\mathcal{B}_n^h$ has a unique highest weight element. Furthermore, if $\mathbf{b}, \mathbf{c} \in \mathcal{B}_n^h$ are highest weight elements such that $\mathrm{wt}(\mathbf{b}) = \mathrm{wt}(\mathbf{c})$, then the connected components generated by $\mathbf{b}$ and $\mathbf{c}$ are isomorphic.*

We denote a connected component with a highest weight element of highest weight λ by $\mathcal{B}_\lambda$. The **character** of the crystal $\mathcal{B}$ is defined to be the polynomial $\chi_{\mathcal{B}}(\mathbf{x}) = \sum_{\mathbf{b} \in \mathcal{B}} \mathbf{x}^{\mathrm{wt}(\mathbf{b})}$ in the variables $\mathbf{x} = (x_1, x_2, \ldots, x_{n+1})$.

Theorem 4 ([9]). *The character of $\mathcal{B}_\lambda$ is equal to the Schur polynomial $s_\lambda(\mathbf{x})$ (or Schur function in the limit $n \to \infty$).*

3 Crystal isomorphism

3.1 Kraśkiewicz insertion

In this section, we describe Kraśkiewicz insertion. To do so, we first need to define the **Edelman–Greene insertion** [5]. It is defined for a word $\mathbf{w} = w_1 \dots w_\ell$ and a letter k such that the concatenation $w_1 \dots w_\ell k$ is an A-type reduced word. The Edelman–Green insertion of a letter k into an *increasing* word $\mathbf{w} = w_1 \dots w_\ell$, denoted by $\mathbf{w} \leftsquigarrow k$, is constructed as follows:

1. If $w_\ell < k$, then $\mathbf{w} \leftsquigarrow k = \mathbf{w}'$, where $\mathbf{w}' = w_1 w_2 \dots w_\ell\, k$.

2. If $k > 0$ and $k\,k+1 = w_i\, w_{i+1}$ for some $1 \leqslant i < \ell$, then $\mathbf{w} \leftsquigarrow k = k+1 \leftsquigarrow \mathbf{w}$.

3. Else let w_i be the leftmost letter in $\mathbf{w}$ such that $w_i > k$. Then $\mathbf{w} \leftsquigarrow k = w_i \leftsquigarrow \mathbf{w}'$, where $\mathbf{w}' = w_1 \dots w_{i-1}\, k\, w_{i+1} \dots w_\ell$.

In the cases above, when $\mathbf{w} \leftsquigarrow k = k' \leftsquigarrow \mathbf{w}'$, the symbol $k' \leftsquigarrow \mathbf{w}'$ indicates a word $\mathbf{w}'$ together with a "bumped" letter k'.

Next we consider a reduced unimodal word $\mathbf{a} = a_1 a_2 \dots a_\ell$ with $a_1 > a_2 > \dots > a_v < a_{v+1} < \dots < a_\ell$. The **Kraśkiewicz row insertion** [10, 11] is defined for a unimodal word $\mathbf{a}$ and a letter k such that the concatenation $a_1 a_2 \dots a_\ell k$ is a C-type reduced word. The Kraśkiewicz row insertion of k into $\mathbf{a}$ (denoted similarly as $\mathbf{a} \leftsquigarrow k$), is performed as follows:

1. If $k = 0$ and there is a subword 101 in $\mathbf{a}$, then $\mathbf{a} \leftsquigarrow 0 = 0 \leftsquigarrow \mathbf{a}$.

2. If $k \neq 0$ or there is no subword 101 in $\mathbf{a}$, denote the decreasing part $a_1 \dots a_v$ as $\mathbf{d}$ and the increasing part $a_{v+1} \dots a_\ell$ as $\mathbf{g}$. Perform the Edelman-Greene insertion of k into $\mathbf{g}$.

   (a) If $a_\ell < k$, then $\mathbf{g} \leftsquigarrow k = a_{v+1} \dots a_\ell k =: \mathbf{g}'$ and $\mathbf{a} \leftsquigarrow k = \mathbf{d}\mathbf{g} \leftsquigarrow k = \mathbf{d}\,\mathbf{g}' =: \mathbf{a}'$.

   (b) If there is a bumped letter and $\mathbf{g} \leftsquigarrow k = k' \leftsquigarrow \mathbf{g}'$, negate all letters in $\mathbf{d}$ (call the resulting word $-\mathbf{d}$) and perform the Edelman-Greene insertion $-\mathbf{d} \leftsquigarrow -k'$. Note that there will always be a bumped letter, and so $-\mathbf{d} \leftsquigarrow -k' = -k'' \leftsquigarrow -\mathbf{d}'$ for some decreasing word $\mathbf{d}'$. The result of the Kraśkiewicz insertion is: $\mathbf{a} \leftsquigarrow k = \mathbf{d}[\mathbf{g} \leftsquigarrow k] = \mathbf{d}[k' \leftsquigarrow \mathbf{g}'] = -[-\mathbf{d} \leftsquigarrow -k']\, \mathbf{g}' = [k'' \leftsquigarrow \mathbf{d}']\mathbf{g}' = k'' \leftsquigarrow \mathbf{a}'$, where $\mathbf{a}' := \mathbf{d}'\mathbf{g}'$.

Example 5. $31012 \leftsquigarrow 0 = 0 \leftsquigarrow 31012, 31012 \leftsquigarrow 1 = 1 \leftsquigarrow 32012.$

The insertion is constructed to "commute" a unimodal word with a letter: If $\mathbf{a} \leftarrow k = k' \leftarrow \mathbf{a}'$, the two elements of the type C Coxeter group corresponding to the words $\mathbf{a}\,k$ and $k'\mathbf{a}'$ are the same.

The type C Stanley symmetric functions (2.1) are defined in terms of unimodal factorizations. To put the formula on a completely combinatorial footing, we need to treat the powers of 2 by introducing signed unimodal factorizations. A **signed unimodal factorization** of $w \in W_C$ is a unimodal factorization $\mathbf{A}$ of w, in which every non-empty factor is assigned either a $+$ or $-$ sign. Denote the set of all signed unimodal factorizations of w by $U^{\pm}(w)$.

For a signed unimodal factorization $\mathbf{A} \in U^{\pm}(w)$, define $\mathrm{wt}(\mathbf{A})$ to be the vector with i-th coordinate equal to the number of letters in the i-th factor of $\mathbf{A}$. Notice from (2.1)

$$F_w^C(\mathbf{x}) = \sum_{\mathbf{A} \in U^{\pm}(w)} \mathbf{x}^{\mathrm{wt}(\mathbf{A})}. \tag{3.1}$$

We will use the Kraśkiewicz insertion to construct a map between signed unimodal factorizations of a Coxeter group element w and pairs of certain types of tableaux $(\mathbf{P}, \mathbf{T})$. We define these types of tableaux next.

A **shifted diagram** $\mathcal{S}(\lambda)$ associated to a partition λ with distinct parts is the set of boxes in positions $\{(i,j) \mid 1 \leqslant i \leqslant \ell(\lambda),\ i \leqslant j \leqslant \lambda_i + i - 1\}$. Here, we use English notation, where the box $(1,1)$ is always top-left.

Let X_n° be an ordered alphabet of n letters $X_n^{\circ} = \{0 < 1 < 2 < \cdots < n-1\}$, and let X_n' be an ordered alphabet of n letters together with their primed counterparts as $X_n' = \{1' < 1 < 2' < 2 < \cdots < n' < n\}$.

Let λ be a partition with distinct parts. A **unimodal tableau P** of shape λ on n letters is a filling of $\mathcal{S}(\lambda)$ with letters from the alphabet X_n° such that the word P_i obtained by reading the ith row from the top of $\mathbf{P}$ from left to right, is a unimodal word, and P_i is the longest unimodal subword in the concatenated word $P_{i+1}P_i$ [2] (cf. also with decomposition tableaux [14, 4]). The **reading word** of a unimodal tableau $\mathbf{P}$ is given by $\pi_{\mathbf{P}} = P_{\ell}P_{\ell-1}\dots P_1$. A unimodal tableau is called *reduced* if $\pi_{\mathbf{P}}$ is a type C reduced word corresponding to the Coxeter group element $w_{\mathbf{P}}$. Given a fixed Coxeter group element w, denote the set of reduced unimodal tableaux $\mathbf{P}$ of shape λ with $w_{\mathbf{P}} = w$ as $\mathcal{UT}_w(\lambda)$.

A **signed primed tableau T** of shape λ on n letters (cf. semistandard Q-tableau [12]) is a filling of $\mathcal{S}(\lambda)$ with letters from the alphabet X_n' such that:

1. The entries are weakly increasing along each column and each row of $\mathbf{T}$.
2. Each row contains at most one i' for every $i = 1, \dots, n$.
3. Each column contains at most one i for every $i = 1, \dots, n$.

The reason for using the word "signed" in the name is to distinguish the set of primed tableaux above from the "unsigned" version described later in the chapter.

Denote the set of signed primed tableaux of shape λ by $\mathcal{PT}^{\pm}(\lambda)$. Given an element $\mathbf{T} \in \mathcal{PT}^{\pm}(\lambda)$, define the weight of the tableau $\mathrm{wt}(\mathbf{T})$ as the vector with i-th coordinate equal to the total number of letters in $\mathbf{T}$ that are either i or i'.

Example 6. $\left(\begin{ytableau} 4 & 3 & 2 & 0 & 1 \\ \none & 2 & 1 & 2 \\ \none & \none & 0 \end{ytableau}\, , \begin{ytableau} 1 & 1 & 2' & 3' & 3 \\ \none & 2' & 2 & 3' \\ \none & \none & 4 \end{ytableau} \right)$ is a pair consisting of a unimodal tableau and a signed primed tableau both of shape $(5, 3, 1)$.

For a reduced unimodal tableau $\mathbf{P}$ with rows $P_\ell, P_{\ell-1}, \ldots, P_1$, the Kraśkiewicz insertion of a letter k into tableau $\mathbf{P}$ (denoted again by $\mathbf{P} \leftsquigarrow k$) is performed as follows:

1. Perform Kraśkiewicz insertion of the letter k into the unimodal word P_1. If there is no bumped letter and $P_1 \leftsquigarrow k = P_1'$, the algorithm terminates and the new tableau $\mathbf{P}'$ consists of rows $P_\ell, P_{\ell-1}, \ldots, P_2, P_1'$. If there is a bumped letter and $P_1 \leftsquigarrow k = k' \leftsquigarrow P_1'$, continue the algorithm by inserting k' into the unimodal word P_2.

2. Repeat the previous step for the rows of $\mathbf{P}$ until either the algorithm terminates, in which case the new tableau $\mathbf{P}'$ consists of rows $P_\ell, \ldots, P_{s+1}, P_s', \ldots, P_1'$, or, the insertion continues until we bump a letter k_e from P_ℓ, in which case we then put k_e on a new row of the shifted shape of $\mathbf{P}'$, so that the resulting tableau $\mathbf{P}'$ consists of rows $k_e, P_\ell', \ldots, P_1'$.

Example 7. $\begin{ytableau} 4 & 3 & 2 & 0 & 1 \\ \none & 2 & 1 & 2 \\ \none & \none & 0 \end{ytableau} \leftsquigarrow 0 = \begin{ytableau} 4 & 3 & 2 & 1 & 0 \\ \none & 2 & 1 & 0 \\ \none & \none & 0 & 1 \end{ytableau}$.

Lemma 1. *[10] Let $\mathbf{P}$ be a reduced unimodal tableau with reading word $\pi_{\mathbf{P}}$ for an element $w \in W_C$. Let k be a letter such that $\pi_{\mathbf{P}} k$ is a reduced word. Then the tableau $\mathbf{P}' = \mathbf{P} \leftsquigarrow k$ is a reduced unimodal tableau, for which the reading word $\pi_{\mathbf{P}'}$ is a reduced word for ws_k.*

Lemma 2. *[12, Lemma 3.17] Let $\mathbf{P}$ be a unimodal tableau, and $\mathbf{a}$ a unimodal word such that $\pi_{\mathbf{P}}\mathbf{a}$ is reduced. Let $(x_1, y_1), \ldots, (x_r, y_r)$ be the (ordered) list of boxes added when $\mathbf{P} \leftsquigarrow \mathbf{a}$ is computed. Then there exists an index v, such that $x_1 < \cdots < x_v \geqslant \cdots \geqslant x_r$ and $y_1 \geqslant \cdots \geqslant y_v < \cdots < y_r$.*

Let $\mathbf{A} \in U^{\pm}(w)$ be a signed unimodal factorization with unimodal factors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$. We recursively construct a sequence $(\varnothing, \varnothing) = (\mathbf{P}_0, \mathbf{T}_0), (\mathbf{P}_1, \mathbf{T}_1), \ldots, (\mathbf{P}_n, \mathbf{T}_n) = (\mathbf{P}, \mathbf{T})$ of tableaux, where $\mathbf{P}_s \in \mathcal{UT}_{(\mathbf{a}_1\mathbf{a}_2\ldots\mathbf{a}_s)}(\lambda^{(s)})$ and $\mathbf{T}_s \in \mathcal{PT}^{\pm}(\lambda^{(s)})$ are tableaux of the same shifted shape $\lambda^{(s)}$.

To obtain the **insertion tableau** $\mathbf{P}_s$, insert the letters of $\mathbf{a}_s$ one by one from left to right, into $\mathbf{P}_{s-1}$. Denote the shifted shape of $\mathbf{P}_s$ by $\lambda^{(s)}$. Enumerate the boxes in the skew shape $\lambda^{(s)}/\lambda^{(s-1)}$ in the order they appear in $\mathbf{P}_s$. Let these boxes be $(x_1, y_1), \ldots, (x_{\ell_s}, y_{\ell_s})$.

Let v be the index that is guaranteed to exist by Lemma 2 when we compute $\mathbf{P}_{s-1} \leftsquigarrow \mathbf{a_s}$. The **recording tableau** $\mathbf{T}_s$ is a primed tableau obtained from $\mathbf{T}_{s-1}$ by adding the boxes $(x_1, y_1), \ldots, (x_{v-1}, y_{v-1})$, each filled with the letter s', and the boxes $(x_{v+1}, y_{v+1}), \ldots, (x_{\ell_s}, y_{\ell_s})$, each filled with the letter s. The special case is the box (x_v, y_v), which could contain either s' or s. The letter is determined by the sign of the factor $\mathbf{a}_s$: If the sign is $-$, the box is filled with the letter s', and if the sign is $+$, the box is filled with the letter s. We call the resulting map the **primed Kraśkiewicz map** KR'.

Example 8. Given a signed unimodal factorization $\mathbf{A} = (-0)(+212)(-43201)$, the sequence of tableaux is

$$(\varnothing, \varnothing), \quad \left(\boxed{0}, \boxed{1'}\right), \quad \left(\begin{array}{|c|c|c|}\hline 2 & 1 & 2 \\ \hline \multicolumn{1}{c|}{} & 0 & \multicolumn{1}{c}{} \\ \cline{2-2}\end{array}, \begin{array}{|c|c|c|}\hline 1' & 2' & 2 \\ \hline \multicolumn{1}{c|}{} & 2 & \multicolumn{1}{c}{} \\ \cline{2-2}\end{array}\right), \quad \left(\begin{array}{|c|c|c|c|c|}\hline 4 & 3 & 2 & 0 & 1 \\ \hline \multicolumn{1}{c|}{} & 2 & 1 & 2 & \multicolumn{1}{c}{} \\ \cline{2-4} \multicolumn{2}{c|}{} & 0 & \multicolumn{2}{c}{} \\ \cline{3-3}\end{array}, \begin{array}{|c|c|c|c|c|}\hline 1' & 2' & 2 & 3' & 3 \\ \hline \multicolumn{1}{c|}{} & 2 & 3' & 3 & \multicolumn{1}{c}{} \\ \cline{2-4} \multicolumn{2}{c|}{} & 3' & \multicolumn{2}{c}{} \\ \cline{3-3}\end{array}\right)$$

If the recording tableau is constructed, instead, by simply labeling its boxes with $1, 2, 3, \ldots$ in the order these boxes appear in the insertion tableau, we recover the original Kraśkiewicz map [10, 11], which is a bijection $\mathrm{KR}\colon R(w) \to \bigcup_\lambda \left[\mathcal{UT}_w(\lambda) \times \mathcal{ST}(\lambda)\right]$, where $\mathcal{ST}(\lambda)$ is the set of **standard shifted tableau** of shape λ, i.e., the set of fillings of $\mathcal{S}(\lambda)$ with letters $1, 2, \ldots, |\lambda|$ such that each letter appears exactly once, each row filling is increasing, and each column filling is increasing.

Theorem 9. *The primed Kraśkiewicz map is a bijection* $\mathrm{KR}'\colon U^{\pm}(w) \to \bigcup_\lambda \left[\mathcal{UT}_w(\lambda) \times \mathcal{PT}^{\pm}(\lambda)\right]$.

Theorem 9 and (3.1) imply the following relation:

$$F_w^C(\mathbf{x}) = \sum_\lambda \left|\mathcal{UT}_w(\lambda)\right| \sum_{\mathbf{T} \in \mathcal{PT}^{\pm}(\lambda)} \mathbf{x}^{\mathrm{wt}(\mathbf{T})}. \tag{3.2}$$

Remark 10. The sum $\sum_{\mathbf{T} \in \mathcal{PT}^{\pm}(\lambda)} \mathbf{x}^{\mathrm{wt}(\mathbf{T})}$ is also known as the Q-Schur function. The expansion (3.2) was shown in [1].

At this point, we are halfway there to expand $F_w^C(\mathbf{x})$ in terms of Schur functions. In the next section we introduce a crystal structure on the set $\mathcal{PT}(\lambda)$ of unsigned primed tableaux.

3.2 Mixed insertion

Set $\mathcal{B}^h = \mathcal{B}^h_\infty$. Similar to the well-known RSK-algorithm, mixed insertion [7] gives a bijection between $\mathcal{B}^h$ and the set of pairs of tableaux $(\mathbf{T}, \mathbf{Q})$, but in this case $\mathbf{T}$ is an (unsigned) primed tableau of shape λ and $\mathbf{Q}$ is a standard shifted tableau of the same shape.

An **(unsigned) primed tableau** of shape λ (cf. semistandard P-tableau [12] or semistandard marked shifted tableau [4]) is a signed primed tableau $\mathbf{T}$ of shape λ with only unprimed elements on the main diagonal. Denote the set of primed tableaux of shape λ by $\mathcal{PT}(\lambda)$. The weight function $\mathrm{wt}(\mathbf{T})$ of $\mathbf{T} \in \mathcal{PT}(\lambda)$ is inherited from the weight function of signed primed tableaux, that is, it is the vector with i-th coordinate equal to the number of letters i' and i in $\mathbf{T}$. We can simplify (3.2) as

$$F_w^C(\mathbf{x}) = \sum_\lambda 2^{\ell(\lambda)} \left|\mathcal{UT}_w(\lambda)\right| \sum_{\mathbf{T} \in \mathcal{PT}(\lambda)} \mathbf{x}^{\mathrm{wt}(\mathbf{T})}. \tag{3.3}$$

Remark 11. The sum $\sum_{\mathbf{T}\in\mathcal{PT}(\lambda)} \mathbf{x}^{\mathrm{wt}(\mathbf{T})}$ is also known as a P-Schur function.

Given any word $b_1 b_2 \dots b_h$ in the alphabet $X = \{1 < 2 < 3 < \cdots\}$, we recursively construct a sequence of tableaux $(\varnothing, \varnothing) = (\mathbf{T}_0, \mathbf{Q}_0), (\mathbf{T}_1, \mathbf{Q}_1), \dots, (\mathbf{T}_h, \mathbf{Q}_h) = (\mathbf{T}, \mathbf{Q})$, where $\mathbf{T}_s \in \mathcal{PT}(\lambda^{(s)})$ and $\mathbf{Q}_s \in \mathcal{ST}(\lambda^{(s)})$. To obtain the tableau $\mathbf{T}_s$, insert the letter b_s into $\mathbf{T}_{s-1}$ as follows. First, insert b_s into the first row of $\mathbf{T}_{s-1}$, bumping out the leftmost element y that is strictly greater than b_i in the alphabet $X' = \{1' < 1 < 2' < 2 < \cdots\}$.

1. If y is not on the main diagonal and y is not primed, then insert it into the next row, bumping out the leftmost element that is strictly greater than y from that row.

2. If y is not on the main diagonal and y is primed, then insert it into the next column to the right, bumping out the topmost element that is strictly greater than y from that column.

3. If y is on the main diagonal, then it must be unprimed. Prime y and insert it into the column on the right, bumping out the topmost element that is strictly greater than y from that column.

If a bumped element exists, treat it as a new y and repeat the steps above – if the new y is unprimed, row-insert it into the row below its original cell, and if the new y is primed, column-insert it into the column to the right of its original cell.

The insertion process terminates either by placing a letter at the end of a row, bumping no new element, or forming a new row with the last bumped element. The shapes of $\mathbf{T}_{s-1}$ and $\mathbf{T}_s$ differ by one box. Add that box to $\mathbf{Q}_{s-1}$ with a letter s in it, to obtain the standard shifted tableau $\mathbf{Q}_s$.

Example 12. For a word 332332123, some of the tableaux in the sequence $(\mathbf{T}_i, \mathbf{Q}_i)$ are

$$\left(\begin{array}{cc} 2 & 3' \\ & 3 \end{array}, \begin{array}{cc} 1 & 2 \\ & 3 \end{array}\right), \quad \left(\begin{array}{cccc} 2 & 2 & 3' & 3 \\ & 3 & 3 & \end{array}, \begin{array}{cccc} 1 & 2 & 4 & 5 \\ & 3 & 6 & \end{array}\right), \quad \left(\begin{array}{ccccc} 1 & 2' & 2 & 3' & 3 \\ & 2 & 3' & 3 & \\ & & 3 & & \end{array}, \begin{array}{ccccc} 1 & 2 & 4 & 5 & 9 \\ & 3 & 6 & 8 & \\ & & 7 & & \end{array}\right).$$

Theorem 13. *[7] The construction above gives a bijection* $\mathrm{HM} \colon \mathcal{B}^h \to \bigcup_{\lambda \vdash h} \left[\mathcal{PT}(\lambda) \times \mathcal{ST}(\lambda)\right]$.

The bijection HM is called a **mixed insertion**. If $\mathrm{HM}(\mathbf{b}) = (\mathbf{T}, \mathbf{Q})$, denote $P_{\mathrm{HM}}(\mathbf{b}) = \mathbf{T}$ and $R_{\mathrm{HM}}(\mathbf{b}) = \mathbf{Q}$. Just as for the RSK-algorithm, the mixed insertion has the property of preserving the recording tableau within each connected component of the crystal $\mathcal{B}^h$.

Theorem 14. *The recording tableau $R_{\mathrm{HM}}(\cdot)$ is constant on each connected component of the crystal $\mathcal{B}^h$.*

Let us fix a recording tableau $\mathbf{Q}_\lambda \in \mathcal{ST}(\lambda)$. Define a map $\Psi_\lambda \colon \mathcal{PT}(\lambda) \to \mathcal{B}^h$ as $\Psi_\lambda(\mathbf{T}) = \mathrm{HM}^{-1}(\mathbf{T}, \mathbf{Q}_\lambda)$. By Theorem 14, the set $\mathrm{Im}(\Psi_\lambda)$ consists of several connected components of $\mathcal{B}^h$. The map Ψ_λ can thus be taken as a crystal isomorphism, and we can define the crystal operators and weight function on $\mathcal{PT}(\lambda)$ as

$$e_i(\mathbf{T}) := (\Psi_\lambda^{-1} \circ e_i \circ \Psi_\lambda)(\mathbf{T}), \quad f_i(\mathbf{T}) := (\Psi_\lambda^{-1} \circ f_i \circ \Psi_\lambda)(\mathbf{T}), \quad \mathrm{wt}(\mathbf{T}) := (\mathrm{wt} \circ \Psi_\lambda)(\mathbf{T}). \tag{3.4}$$

Although it is not clear that the crystal operators constructed above are independent of the choice of $\mathbf{Q}_\lambda$, in the next section we will construct explicit crystal operators on the set $\mathcal{PT}(\lambda)$ that satisfy the relations above and do not depend on the choice of $\mathbf{Q}_\lambda$.

Example 15. For $\mathbf{T} = \begin{array}{ccccc} 1 & 2' & 2 & 3' & 3 \\ & 2 & 3' & 3 & \\ & & 3 & & \end{array}$, choose $\mathbf{Q}_\lambda = \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ & 6 & 7 & 8 & \\ & & 9 & & \end{array}$. Then $\Psi_\lambda(\mathbf{T}) = 333332221$ and $e_1 \circ \Psi_\lambda(\mathbf{T}) = 333331221$. Thus, $e_1(\mathbf{T}) = (\Psi_\lambda^{-1} \circ e_1 \circ \Psi_\lambda)(\mathbf{T}) = \begin{array}{ccccc} 1 & 1 & 2 & 3' & 3 \\ & 2 & 3' & 3 & \\ & & 3 & & \end{array}$, $f_1(\mathbf{T}) = f_2(\mathbf{T}) = \mathbf{0}$.

To summarize, we obtain a crystal isomorphism between the crystal $(\mathcal{PT}(\lambda), e_i, f_i, \mathrm{wt})$, denoted again by $\mathcal{PT}(\lambda)$, and a direct sum $\bigoplus_\mu \mathcal{B}_\mu^{\oplus h_{\lambda\mu}}$. We will provide a combinatorial description of the coefficients $h_{\lambda\mu}$ in the next section. This implies the relation on characters of the corresponding crystals $\chi_{\mathcal{PT}(\lambda)} = \sum_\mu h_{\lambda\mu} s_\mu$. Thus we can rewrite (3.3) one last time

$$F_w^C(\mathbf{x}) = \sum_\lambda 2^{\ell(\lambda)} |\mathcal{UT}_w(\lambda)| \sum_\mu h_{\lambda\mu} s_\mu = \sum_\mu \Big(\sum_\lambda 2^{\ell(\lambda)} |\mathcal{UT}_w(\lambda)| \; h_{\lambda\mu} \Big) s_\mu.$$

3.3 Explicit crystal operators on shifted primed tableaux

We consider the alphabet $X' = \{1' < 1 < 2' < 2 < 3' < \cdots\}$ of primed and unprimed letters. It is useful to think about the letter $(i+1)'$ as a number $i + 0.5$. Thus, we say that letters i and $(i+1)'$ differ by half a unit and letters i and $(i+1)$ differ by a whole unit.

Given an (unsigned) primed tableau $\mathbf{T}$, the **reading word** $\mathrm{rw}(\mathbf{T})$ is constructed as:

1. List all primed letters in the tableau, column by column, in decreasing order within each column, moving from the rightmost column to the left, and with all the primes removed (i.e. all letters are increased by half a unit). (Call this part of the word the **primed reading word**.)

2. Then list all unprimed elements, row by row, in increasing order within each row, moving from the bottommost row to the top. (Call this part of the word the **unprimed reading word**.)

To find the letter on which the crystal operator f_i acts, apply the bracketing rule for letters i and $i+1$ within the reading word $\mathrm{rw}(\mathbf{T})$. If all letters i are bracketed in $\mathrm{rw}(\mathbf{T})$, then $f_i(\mathbf{T}) = \mathbf{0}$. Otherwise, the rightmost unbracketed letter i in $\mathrm{rw}(\mathbf{T})$ corresponds to an i or an i' in $\mathbf{T}$, which we call **bold unprimed** i or **bold primed** i respectively. If the bold letter i is unprimed, denote the cell it is located in as x. If the bold letter i is primed, we *conjugate* the tableau $\mathbf{T}$ first.

The **conjugate** of a primed tableau $\mathbf{T}$ is obtained by reflecting the tableau over the main diagonal, changing all primed entries k' to k and changing all unprimed elements k to $(k+1)'$ (i.e. increase the content of all boxes by half a unit). The main diagonal is now the North-East boundary of the tableau. Denote the resulting tableau as $\mathbf{T}^*$.

Under the transformation $\mathbf{T} \to \mathbf{T}^*$, the bold primed i is transformed into bold unprimed i. Denote the cell it is located in as x.

Given any cell z in a shifted primed tableau $\mathbf{T}$ (or conjugated tableau $\mathbf{T}^*$), denote by $c(z)$ the content of z. Denote by z_E the cell to the right of z, z_W the cell to its left, z_S the cell below, and z_N the cell above. Denote by z^* the corresponding conjugated cell in $\mathbf{T}^*$ (or in $\mathbf{T}$). Now, consider the box x_E (in $\mathbf{T}$ or in $\mathbf{T}^*$) and notice that $c(x_E) \geqslant (i+1)'$.

Crystal operator f_i on primed tableaux:

1. If $c(x_E) = (i+1)'$, the box x must lie outside of the main diagonal and the box right below x_E cannot have content equal to $(i+1)'$. Change $c(x)$ to $(i+1)'$ and change $c(x_E)$ to $(i+1)$ (i.e. increase the content of x and x_E by half a unit).
2. If $c(x_E) \neq (i+1)'$ or x_E is empty, then there is a maximal connected ribbon (expanding in South and West directions) with the following properties:
   (a) The North-Eastern most box of the ribbon (the tail of the ribbon) is x.
   (b) The contents of all boxes within a ribbon besides the tail are either $(i+1)'$ or $(i+1)$.

   Denote the South-Western most box of the ribbon (the head) as x_H.

   (a) If $x_H = x$, change $c(x)$ to $(i+1)$ (i.e. increase the content of x by a whole unit).
   (b) If $x_H \neq x$ and x_H is on the main diagonal (in case of a tableau $\mathbf{T}$), change $c(x)$ to $(i+1)'$ (i.e. increase the content of x by half a unit).
   (c) Otherwise, the content $c(x_H)$ must be $(i+1)'$ due to the bracketing rule. We change $c(x)$ to $(i+1)'$ and change $c(x_H)$ to $(i+1)$ (i.e. increase the content of x and x_H by half a unit).

In the case when the bold i in $\mathbf{T}$ is unprimed, we apply the above crystal operator rules to $\mathbf{T}$ to find $f_i(\mathbf{T})$

Example 16. In the following examples, we mark the bold i (if it exists):

$$f_2\left(\begin{array}{|c|c|c|c|}\hline 1&2'&2&3'\\ \hline \multicolumn{1}{c|}{}&2&3'&3\\ \cline{2-4}\end{array}\right)=\mathbf{0},\quad f_2\left(\begin{array}{|c|c|c|c|}\hline 1&2'&\mathbf{2}&3'\\ \hline \multicolumn{1}{c|}{}&2&3'&4\\ \cline{2-4}\end{array}\right)=\begin{array}{|c|c|c|c|}\hline 1&2'&3'&3\\ \hline \multicolumn{1}{c|}{}&2&3'&4\\ \cline{2-4}\end{array},\quad f_2\left(\begin{array}{|c|c|c|c|}\hline 1&1&2&\mathbf{2}\\ \hline \multicolumn{1}{c|}{}&3&4'&4\\ \cline{2-4}\end{array}\right)=\begin{array}{|c|c|c|c|}\hline 1&1&2&3\\ \hline \multicolumn{1}{c|}{}&3&4'&4\\ \cline{2-4}\end{array},$$

$$f_2\left(\begin{array}{|c|c|c|c|c|}\hline 1&1&2'&\mathbf{2}&3\\ \hline \multicolumn{1}{c|}{}&2&2&3'&\multicolumn{1}{c}{}\\ \cline{2-4}\multicolumn{2}{c|}{}&3&3&\multicolumn{1}{c}{}\\ \cline{3-4}\end{array}\right)=\begin{array}{|c|c|c|c|c|}\hline 1&1&2'&3'&3\\ \hline \multicolumn{1}{c|}{}&2&2&3'&\multicolumn{1}{c}{}\\ \cline{2-4}\multicolumn{2}{c|}{}&3&3&\multicolumn{1}{c}{}\\ \cline{3-4}\end{array},\quad f_2\left(\begin{array}{|c|c|c|c|c|}\hline 1&1&1&\mathbf{2}&3\\ \hline \multicolumn{1}{c|}{}&2&2&3'&\multicolumn{1}{c}{}\\ \cline{2-4}\multicolumn{2}{c|}{}&3&4'&\multicolumn{1}{c}{}\\ \cline{3-4}\end{array}\right)=\begin{array}{|c|c|c|c|c|}\hline 1&1&1&3'&3\\ \hline \multicolumn{1}{c|}{}&2&2&3&\multicolumn{1}{c}{}\\ \cline{2-4}\multicolumn{2}{c|}{}&3&4'&\multicolumn{1}{c}{}\\ \cline{3-4}\end{array}.$$

In the case when the bold i is primed in $\mathbf{T}$, we first conjugate $\mathbf{T}$ and then apply the above crystal operator rules on $\mathbf{T}^*$, before reversing the conjugation. Note that Case 2b is impossible for $\mathbf{T}^*$, since the main diagonal is now on the North-East.

Example 17. Let $\mathbf{T} = \begin{array}{|c|c|c|c|}\hline 1&\mathbf{2'}&2&3\\ \hline \multicolumn{1}{c|}{}&3&4'&\multicolumn{1}{c}{}\\ \cline{2-3}\multicolumn{2}{c|}{}&4&\multicolumn{1}{c}{}\\ \cline{3-3}\end{array}$, then $\mathbf{T}^* = \begin{array}{|c|c|c|}\cline{1-1} 2'&\multicolumn{2}{c}{}\\ \cline{1-2}\mathbf{2}&4'&\multicolumn{1}{c}{}\\ \hline 3'&4&5'\\ \hline 4'&\multicolumn{2}{c}{}\\ \cline{1-1}\end{array}$ and $f_2(\mathbf{T}) = \begin{array}{|c|c|c|c|}\hline 1&2&3'&3\\ \hline \multicolumn{1}{c|}{}&3&4'&\multicolumn{1}{c}{}\\ \cline{2-3}\multicolumn{2}{c|}{}&4&\multicolumn{1}{c}{}\\ \cline{3-3}\end{array}$.

Theorem 18. *For any $\mathbf{b} \in \mathcal{B}^h$ with $P_{\mathrm{HM}}(\mathbf{b}) = \mathbf{T}$ and $f_i(\mathbf{b}) \neq \mathbf{0}$, the operator f_i defined on above satisfies*

$$P_{\mathrm{HM}}(f_i(\mathbf{b})) = f_i(\mathbf{T}).$$

Also, $f_i(\mathbf{b}) = \mathbf{0}$ if and only if $f_i(\mathbf{T}) = \mathbf{0}$.

The crystal operators $e_i(\mathbf{T})$ are defined similarly. Consider the reading word $\mathrm{rw}(\mathbf{T})$ and apply the bracketing rule on the letters i and $i+1$. If all letters $i+1$ are bracketed in $\mathrm{rw}(\mathbf{T})$, then $e_i(\mathbf{T}) = \mathbf{0}$. Otherwise, the action of e_i on $\mathbf{T}$ can be obtained from the action of f_i on $-\mathbf{T}$. For more details we refer to the long version of the paper.

Theorem 19. *Given a primed tableau $\mathbf{T}$ with $f_i(\mathbf{T}) \neq \mathbf{0}$, the operators e_i satisfy $e_i(f_i(\mathbf{T})) = \mathbf{T}$.*

Corollary 20. *For any $\mathbf{b} \in \mathcal{B}^h$ with $\mathrm{HM}(\mathbf{b}) = (\mathbf{T}, \mathbf{Q})$, the operator e_i defined above satisfies $\mathrm{HM}(e_i(\mathbf{b})) = (e_i(\mathbf{T}), \mathbf{Q})$, given the left-hand side is well-defined.*

The consequence of Theorem 18, as discussed in Section 3.2, is a crystal isomorphism $\Psi_\lambda \colon \mathcal{PT}(\lambda) \to \bigoplus \mathcal{B}_\mu^{\oplus h_{\lambda\mu}}$. Now, to determine the nonnegative integer coefficients $h_{\lambda\mu}$, it is enough to count the highest weight elements in $\mathcal{PT}(\lambda)$ of given weight μ.

Proposition 21. *A primed tableau $\mathbf{T} \in \mathcal{PT}(\lambda)$ is a highest weight element if and only if its reading word $\mathrm{rw}(\mathbf{T})$ is a Yamanouchi word. That is, for any suffix of $\mathrm{rw}(\mathbf{T})$, its weight is a partition.*

Thus we define $h_{\lambda\mu}$ to be the number of primed tableaux $\mathbf{T}$ of shifted shape $\mathcal{S}(\lambda)$ and weight μ such that $\mathrm{rw}(\mathbf{T})$ is Yamanouchi.

Example 22. Let $\lambda = (5,3,2)$ and $\mu = (4,3,2,1)$. There are three primed tableaux of shifted shape $\mathcal{S}((5,3,2))$ and weight $(4,3,2,1)$ with a Yamanouchi reading word, namely $\begin{array}{ccccc} 1 & 1 & 1 & 1 & 2' \\ & 2 & 2 & 3' & \\ & & 3 & 4' & \end{array}$, $\begin{array}{ccccc} 1 & 1 & 1 & 1 & 3' \\ & 2 & 2 & 2 & \\ & & 3 & 4' & \end{array}$ and $\begin{array}{ccccc} 1 & 1 & 1 & 1 & 4' \\ & 2 & 2 & 2 & \\ & & 3 & 3 & \end{array}$. Therefore $h_{(5,3,2)(4,3,2,1)} = 3$.

We summarize our results for the type C Stanley symmetric functions as follows.

Corollary 23. *The expansion of $F_w^C(\mathbf{x})$ in terms of Schur symmetric functions is*

$$F_w^C(\mathbf{x}) = \sum_\lambda g_{w\lambda} s_\lambda(\mathbf{x}), \quad \text{where} \quad g_{w\lambda} = \sum_\mu 2^{\ell(\mu)} \left|\mathcal{UT}_w(\mu)\right| h_{\mu\lambda}\,. \tag{3.5}$$

Example 24. Consider the word $w = 0101 = 1010$. There is only one unimodal tableau corresponding to w, namely $\mathbf{P} = \begin{array}{ccc} 1 & 0 & 1 \\ & 0 & \end{array}$, which belongs to $\mathcal{UT}_{0101}(3,1)$. Thus, $g_{w\lambda} = 4h_{(3,1)\lambda}$. There are only three possible highest weight primed tableaux of shape $(3,1)$, namely $\begin{array}{ccc} 1 & 1 & 1 \\ & 2 & \end{array}$, $\begin{array}{ccc} 1 & 1 & 2' \\ & 2 & \end{array}$ and $\begin{array}{ccc} 1 & 1 & 3' \\ & 2 & \end{array}$, which implies that $h_{(3,1)(3,1)} = h_{(3,1)(2,2)} = h_{(3,1)(2,1,1)} = 1$ and $h_{(3,1)\lambda} = 0$ for other weights λ. The expansion of $F_{0101}^C(\mathbf{x})$ is thus

$$F_{0101}^C = 4s_{(3,1)} + 4s_{(2,2)} + 4s_{(2,1,1)}.$$

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